On the Solutions of Certain Types of Linear Differential Equations with Periodic Coefficients.

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§ 1. *Introduction.*

Most of the methods which are employed for finding the solutions of differential equations were devised in order to solve the practical problems which arise in celestial mechanics. It is sufficient to mention in this connection the expansion of the solutions as power series in the independent variable, as power series in parameters, the method of the variation of parameters in the general non-linear case, and the method of successive approximations. The first two and the last were used formally by the founders of the analytic theory of the motions of the planets—Clairaut, d'Alembert, and Euler—and the third was given its general formulation by Lagrange at the end of the eighteenth century, and its widest application by Delaunay in his Lunar Theory, in the middle of the nineteenth century. All of these processes were extensively employed in celestial mechanics for obtaining practical results, without any inquiry being made regarding the circumstances and realm of their validity. Indeed, it was as late as 1842 that Cauchy† began laying the foundations of the modern theories of differential equations.

The recent contributions to the theory of differential equations, which have been stimulated by problems in celestial mechanics, have come chiefly from the hands of Hill and Poincaré. In 1877 Hill‡ published privately at Cambridge, Mass., his famous investigation of the motion of the lunar perigee. In this memoir he treated with rare skill the differential equation

$$\frac{d^2w}{dt^2} + \theta(t)w = 0,$$

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†Cauchy's *Collected Works*, 3d series, Vol. VII.

‡Also reprinted in *Acta Mathematica*, Vol. VIII (1886), pp. 1–36; Hill's *Collected Works*, Vol. I, pp. 243–270.

where θ is a simply periodic function of t. About the same time Hermite* discovered the form of the solution of Lamé's equation, which has a doubly periodic coefficient. Starting from Hermite's results Picard† showed that *in general* a fundamental set of solutions of a linear differential equation of the n-th order having doubly periodic coefficients of the first kind can be expressed in terms of doubly periodic functions of the second kind. In 1883 Floquet‡ published a complete discussion of the character of the solutions of a homogeneous linear differential equation of the n-th order having simply periodic coefficients. In this memoir Floquet gave not only the form of the solution in general, but he considered in detail the forms of the solutions when the fundamental equation has multiple roots. The forms of the solutions being thus known, the efforts of later writers have been directed toward the discovery of practical means for their actual construction. Among those who have discussed the problem of finding the solutions of Hill's equation we may mention Lindemann§, Lindstedt||, Bruns¶, Callandreau**, Stieltjes††, and Harzer‡‡.

In all of these investigations a large amount of attention has been devoted to finding the roots of the fundamental equation, or equivalent transcendentals. Hill determined them from an infinite determinant which he first introduced into analysis in this connection; Lindstedt found them from an infinite continued fraction§§. In all cases these transcendentals were computed first, and then the solutions were found later. It should be noted also that the processes are valid only under certain special conditions which, fortunately, are satisfied in the case of Hill's equation. The problem is treated in a much more general way in Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I, Chapter II, Sec. 29, and Chapter IV.

In dynamical problems involving accelerations simultaneous differential equations of the second order naturally arise, but it is easy to reduce them to twice the number of simultaneous equations of the first order. Now n simul-

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* *Comptes Rendus*, 1877 et seq.

† *Comtes Rendus*, 1879–1880; *Journal für Mathematik*, Vol. XC (1881).

‡ *Annales de l'École Normale Supérieure*, 1883–1884.

§ *Mathematische Annalen*, Vol. XXII (1883), p. 117.

|| *Astronomische Nachrichten*, No. 2503 (1883), and *Mémoires de l'Académie de St. Pétersbourg*, Vol. XXI, No. 4.

¶ *Astronomische Nachrichten*, Nos. 2533 and 2553 (1883).

** *Ibid.*, No. 2547 (1883).

†† *Ibid.*, Nos. 2601 and 2609 (1884).

‡‡ *Ibid.*, Nos. 2850 and 2851 (1888).

§§ See Tisserand's *Mécanique Céleste*, Vol. III, Chapter I.

taneous differential equations of the first order include one differential equation of the n-th order, for the latter can always be reduced to the former. Hence we shall treat here as the general case and the one most simply connecting with dynamical problems a set of simultaneous linear differential equations of the first order having simply periodic coefficients. We shall find the character of the solutions of the differential equations without further restrictions by a very direct process. Then, simple and convenient methods are given for constructing the solutions in all cases in which the coefficients of the differential equations are expansible as power series in a parameter μ, and the terms not depending upon μ (at least in a large part of the discussion) are constants with respect to the independent variable. The linear differential equations having periodic coefficients which arise in celestial mechanics,* of which Hill's equation is a simple example, belong to this class.

§ 2. *The Fundamental Equation.*

We shall consider the differential equations

$$x_i' = \sum_{j=1}^{n} \theta_{ij}(t) x_j, \qquad i = 1, \ldots, n, \tag{1}$$

where x_i' is the derivative of x_i with respect to the independent variable t, and where the θ_{ij} are uniform analytic functions of t and are periodic with the period 2π. Let

$$x_{i1} = \phi_{i1}(t), \quad \ldots, \quad x_{in} = \phi_{in}(t), \qquad i = 1, \ldots, n,$$

be a fundamental set of solutions of (1), where $x_{ij} = \phi_{ij}(t)$, $i = 1, \ldots, n$, is the j-th solution. The determinant of the fundamental set,

$$\Delta = |\phi_{ij}|,$$

is found, by taking the derivative and reducing by means of (1), to satisfy the relation

$$\Delta' = \Delta \sum_{i=1}^{n} \theta_{ii}(t);$$

whence †

$$\Delta = \Delta_0 e^{\int_{t_0}^{t} \sum_{i=1}^{n} \theta_{ii}\, dt} \tag{2}$$

Hence Δ can become zero or infinite only at a singularity of some $\theta_{ii}(t)$.

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*Some of the methods exhibited here were devised in connection with problems raised in the theory of periodic orbits. They were first applied to Hill's equation by Moulton in a paper whose abstract is in *Bull. of the Am. Math. Soc.*, Vol. XIII (1906–7), p. 71, and were later extended by our joint investigations to the general case.

† Darboux, *Comptes Rendus*, Vol. XC (1880), p. 526.

We now start from a particular set of solutions satisfying the initial conditions

$$\phi_{ii}(0)=1, \quad \phi_{ij}(0)=0, \qquad \text{if } j \neq i. \tag{3}$$

It is clear that a set of n solutions satisfying these initial conditions can be constructed, and since for them $\Delta_0 = 1$ is distinct from zero, they form a *fundamental* set of solutions.

Let us make the transformation

$$x_i = e^{at} y_i, \tag{4}$$

α being an undetermined constant. Then equations (1) become

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \theta_{ij} y_j. \tag{5}$$

Any solution of (5) can be written in the form

$$y_i = e^{-\alpha t} \sum_{j=1}^{n} A_j \phi_{ij}(t), \qquad i = 1, \ldots, n, \tag{6}$$

where the A_j are suitably chosen constants.

We now inquire whether it is possible to determine α and the A_j so that the y_i, as defined by (6), shall be periodic with the period 2π. From the form of (5) it is clear that sufficient conditions for the periodicity of the y_i with the period 2π are

$$y_i(2\pi) - y_i(0) = 0, \qquad i = 1, \ldots, n. \tag{7}$$

Imposing these conditions on (6), we get

$$\sum_{j=1}^{n} A_j[\phi_{ij}(2\pi) - e^{2\alpha\pi}\phi_{ij}(0)] = 0, \qquad i = 1, \ldots, n. \tag{8}$$

In order that these equations may have a solution other than $A_1 = \ldots = A_n = 0$, the determinant of the coefficients of the A_j must equal zero. Making use of (3), representing $\phi_{ij}(2\pi)$ simply by ϕ_{ij}, and letting $e^{2\alpha\pi} = s$, the determinant is

$$D = \begin{vmatrix} \phi_{11} - s, & \phi_{12}, & \ldots, & \phi_{1n} \\ \phi_{21}, & \phi_{22} - s, & \ldots, & \phi_{2n} \\ \ldots\ldots, & \ldots\ldots, & \ldots, & \ldots \\ \phi_{n1}, & \phi_{n2}, & \ldots, & \phi_{nn} - s \end{vmatrix} = 0. \tag{9}$$

This is the *fundamental equation* associated with the period 2π and is of degree n in s. It admits neither $s = 0$ nor $s = \infty$ as a root, since the absolute term is the value of the determinant of the fundamental set of solutions at the regular point $t = 2\pi$, and the coefficient of s^n is $(-1)^n$.

§ 3. *Form of the Solutions.*

Suppose the roots of (9) are $s_1, \ldots, s_n$ and that they are all distinct. For each of them there is therefore at least one first minor of D which is distinct from zero, and hence for each of them the ratios of the A_j can be determined from (8) so that the y_i shall be periodic. In this way n solutions are obtained which can be shown to constitute a fundamental set.

Suppose now that $s_2 = s_1$ and that all the other roots of (9) are distinct. There are two cases to be considered, according as all of the first minors of D vanish or do not vanish for $s = s_1$. Suppose all the first minors of D vanish for $s = s_1$; since by hypothesis $s = s_1$ is a double root and not a triple root, there is at least one second minor of D which does not vanish for $s = s_1$. Therefore two of the A_j can be taken arbitrarily and the remaining $n - 2$ can be expressed in terms of them so that the y_i shall be periodic. We thus obtain two distinct solutions of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1} \quad \text{and} \quad x_{i2} = e^{\alpha_1 t} y_{i2}, \qquad i = 1, \ldots, n, \tag{10}$$

where the y_{i1} and y_{i2} are periodic.

Suppose $s = s_1$ is a double root of (9) and that not all its first minors vanish for $s = s_1$. Then there is but one solution of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1}.$$

The y_{i1} are expressible linearly in terms of the ϕ_{ij} by (6). Let the notation be chosen so that the minor which is not zero is formed from the elements of the last $n - 1$ columns. Then A_1 must be distinct from zero in order to avoid the trivial case in which all the A_j are zero.

Then we take as a new set of solutions

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \quad x_{ij} = \phi_{ij}(t), \qquad i = 1, \ldots, n; \; j = 2, \ldots, n. \tag{11}$$

These solutions constitute a fundamental set, for their determinant is

$$e^{\alpha_1 t} \begin{vmatrix} y_{11}, & \phi_{12}, & \ldots, & \phi_{1n} \\ y_{21}, & \phi_{22}, & \ldots, & \phi_{2n} \\ \ldots, & \ldots, & \ldots, & \ldots \\ y_{n1}, & \phi_{n2}, & \ldots, & \phi_{nn} \end{vmatrix},$$

which becomes, by means of (6), $A_1 |\phi_{ij}|$, which is distinct from zero. Starting with this fundamental set, the fundamental equation is

$$D = (s - s_1)\, D_1 = (s - s_1) \begin{vmatrix} y_{11}(0), & \phi_{12}, & \ldots, & \phi_{1n} \\ y_{21}(0), & \phi_{22} - s, & \ldots, & \phi_{2n} \\ \ldots, & \ldots, & \ldots, & \ldots \\ y_{n1}(0), & \phi_{n2}, & \ldots, & \phi_{nn} - s \end{vmatrix} = 0. \qquad (12)$$

Since $s = s_1$ is a double root of D, the determinant D_1 has a single factor $s - s_1$. Since (11) constitute a fundamental set, any solution can be expressed in the form

$$x_i = B_1 e^{\alpha_1 t} y_{i1} + \sum_{j=2}^{n} B_j \phi_{ij}, \qquad i = 1, \ldots, n. \qquad (13)$$

Now we make the transformation, corresponding to (4), to get a second solution associated with α_1,

$$x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1}). \qquad (14)$$

Imposing the condition that the x_{i2} shall satisfy (1) we find, since $e^{\alpha_1 t} y_{i1}$ is a solution,

$$y'_{i2} + \alpha_1 y_{i2} = \sum_{j=1}^{n} \theta_{ij} y_{j2} - y_{i1}, \qquad i = 1, \ldots, n.$$

Therefore sufficient conditions that the y_{i2} shall be periodic with the period 2π are

$$y_{i2}(2\pi) - y_{i2}(0) = 0 = -2\pi y_{i1}(0) + \sum_{j=2}^{n} B_j [e^{-2\alpha_1 \pi} \phi_{ij}(2\pi) - \phi_{ij}(0)].$$

Substituting s_1 for $e^{2\alpha_1 \pi}$, we have

$$-2\pi s_1 y_{i1}(0) + \sum_{j=2}^{n} B_j [\phi_{ij}(2\pi) - s_1 \phi_{ij}(0)] = 0, \qquad i = 1, \ldots, n. \qquad (15)$$

The condition that these equations shall be consistent is

$$D_1 = \begin{vmatrix} y_{11}(0), & \phi_{12}, & \ldots, & \phi_{1n} \\ y_{21}(0), & \phi_{22} - s_1, & \ldots, & \phi_{2n} \\ \ldots, & \ldots, & \ldots, & \ldots \\ y_{n1}(0), & \phi_{n2}, & \ldots, & \phi_{nn} - s_1 \end{vmatrix} = 0.$$

It was shown in (12) that this determinant has a single root $s = s_1$. Hence not all its first minors are zero. By hypothesis not all minors of the first order formed from the last $n - 1$ columns of D, which are the same as the last $n - 1$ columns of D_1, are zero. Therefore we can solve equations (15) for $B_2, \ldots, B_n$ in terms of $y_{i1}(0)$ which carry an arbitrary constant as a factor. Consequently in this case a second solution depending upon α_1 exists and is expressible in the form (14), where y_{i1} and y_{i2} are both periodic.

When $s = s_1$ is a triple root of $D = 0$, an analogous discussion shows that the three associated solutions are of the form

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \quad x_{i2} = e^{\alpha_1 t} y_{i2}, \qquad x_{i3} = e^{\alpha_1 t} y_{i3}, \qquad i = 1, \ldots, n,$$

or

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \quad x_{i2} = e^{\alpha_1 t} y_{i2}, \qquad x_{i3} = e^{\alpha_1 t}[y_{i3} + t(y_{i1} + y_{i2})], \quad i = 1, \ldots, n,$$

or

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \quad x_{i2} = e^{\alpha_1 t}[y_{i2} + t y_{i1}], \quad x_{i3} = e^{\alpha_1 t}[y_{i3} + t y_{i2} + \tfrac{1}{2} t^2 y_{i1}], \quad i = 1, \ldots, n,$$

according as all the minors of D of the first and second orders vanish for $s = s_1$, or all those of the first order but not all those of the second order vanish, or not all the minors of the first order vanish.

In case $s = s_1$ is a root of multiplicity ν, the associated group of solutions have the form

$$\begin{aligned} x_{i1} &= e^{\alpha_1 t} y_{i1}, \\ x_{i2} &= e^{\alpha_1 t}[y_{i2} + t y_{i1}], \\ x_{i3} &= e^{\alpha_1 t}[y_{i3} + t y_{i2} + \tfrac{1}{2} t^2 y_{i3}], \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ x_{i\nu} &= e^{\alpha_1 t}\left[[y_{i\nu} + t y_{i(\nu-1)} + \cdots + \frac{1}{(\nu-1)!} t^{\nu-1} y_{i1}\right], \end{aligned} \tag{16}$$

where the y_{ij} are periodic with the period 2π, provided not all the minors of D of the first order vanish for $s = s_1$. In case all the minors of D of order $k-1$, but not all of order k, vanish for $s = s_1$, there are k solutions of the type of the first of (16), the others involving products of t to powers not exceeding $\nu - k$ and the y_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, \nu$. The details for a single differential equation of order n were given by Floquet (*loc. cit.*), and the results are similar here.

§ 4. *The Characteristic Equation when the Coefficients, θ_{ij}, are Power Series in a Parameter μ.*

We shall now assume that the θ_{ij} are expansible as power series in a parameter μ whose coefficients are separately periodic with the period 2π, and that the series converge for all real finite values of t if $|\mu| < \rho$. We shall assume further that for $\mu = 0$, $\theta_{ij} = a_{ij}$, where the a_{ij} are constants. Under these conditions, which are often realized in practice, particularly in celestial mechanics,*

* Hill's differential equation, considered first in his paper on the motion of the lunar perigee, is an equation of this type.

the discussion of the character of the solutions can be made so as to lead to a convenient method for their explicit construction.

Consider the equations

$$x_i' = \sum_{j=1}^{n} \theta_{ij} x_j = \sum_{j=1}^{n} [a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k] x_j, \qquad i = 1, \ldots, n, \tag{17}$$

where the a_{ij} are constants. When $\mu = 0$ they have the solution

$$x_i^{(0)} = c_i e^{\alpha^{(0)} t},$$

where $\alpha^{(0)}$ is any one of the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - \alpha^{(0)}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22} - \alpha^{(0)}, & \ldots, & a_{2n} \\ \ldots, & \ldots, & \ldots, & \ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} - \alpha^{(0)} \end{vmatrix} = 0. \tag{18}$$

If the n roots of this equation are distinct, a fundamental set of solutions of (17) for $\mu = 0$ is

$$x_{ij}^{(0)} = c_{ij} e^{\alpha_j^{(0)} t}, \qquad i, j = 1, \ldots, n,$$

where the c_{ij} can be taken so that $|c_{ij}| = 1$.

By Poincaré's extension of Cauchy's theorem,* equations (17) can be integrated as power series in μ which will converge for $0 \overline{\gtrless} t \leqq T$, T any arbitrary finite value, provided $|\mu| < R$, R depending upon T. Hence we can write the general solution of (17) in the form

$$x_i = \sum_{j=1}^{n} A_j [c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k], \tag{19}$$

where the A_j are the constants of integration, and the $x_{ij}^{(k)}$ are functions of t depending on the θ_{ij}. We can take the initial conditions so that

$$x_{ij}(0) = \sum_{k=0}^{\infty} x_{ij}^{(k)}(0) \mu^k \equiv c_{ij}.$$

Therefore

$$x_{ij}^{(0)}(0) = c_{ij}, \quad x_{ij}^{(k)}(0) = 0, \qquad k = 1, \ldots, \infty.$$

Now we make the transformation

$$x_i = e^{\alpha t} y_i,$$

after which the differential equations and their solutions become

* *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I, Chapter II.

$$\left.\begin{aligned} y_i' + \alpha y_i &= \sum_{j=1}^{n} [a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k] y_i, & i = 1, \ldots, n; \\ y_i &= \sum_{j=1}^{n} A_j e^{-\alpha t} [c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)}(t) \mu^k], & i = 1, \ldots, n. \end{aligned}\right\} \qquad (20)$$

Imposing the conditions that the y_i shall be periodic with the period 2π, viz., $y_i(2\pi) - y_i(0) = 0$, we get

$$0 = \sum_{j=1}^{n} A_j [c_{ij}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k], \qquad i = 1, \ldots, n. \qquad (21)$$

In order not to have the trivial case where all the A_j are zero, we must set the determinant

$$D = | [c_{ij}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k] | = 0, \qquad (22)$$

which is a condition upon the undetermined constant α. This equation has an infinite number of solutions; for if it is satisfied by $\alpha = \alpha_0$, it is also satisfied by $\alpha = \alpha_0 + \nu\pi\sqrt{-1}$, where ν is any integer. All distinct solutions can be obtained with ν equal to zero; the others amount simply to taking periodic factors from the y_i. The fundamental equation corresponding to (22) is obtained by putting $e^{2\alpha\pi} = s$. If the n values of s satisfying the fundamental equation are distinct, the corresponding values of α are distinct, but not necessarily the converse. We shall use only those n values of α which reduce to the $\alpha_j^{(0)}$ for $\mu = 0$, the n $\alpha_j^{(0)}$ being uniquely determined by (18).

In the event that two of the roots of the characteristic equation, say $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, are equal, the solutions are either of the form (19), where $\alpha_2^{(0)} = \alpha_1^{(0)}$, or

$$\begin{aligned} x_i = A_1 [c_{i1} e^{\alpha_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i1} \mu^k] + A_2 [(c_{i2} + t c_{i1}) e^{\alpha_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i2}^{(k)} \mu^k] \\ + \sum_{j=3}^{n} A_j [c_{ij} e^{\alpha_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k]. \end{aligned} \qquad (23)$$

After making the transformation $x_i = e^{\alpha t} y_i$, writing out the solutions for y_i from (23), and imposing the periodicity conditions, $y_i(2\pi) - y_i(0) = 0$, on the y_i, we find that if not all the A_j are zero either (22) is satisfied or

$$\begin{aligned} D = | &[c_{i1}(e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{i1}^{(k)}(2\pi) \mu^k], \\ &[c_{i2}(e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1) + 2\pi c_{i1} e^{2(\alpha_1^{(0)} - \alpha)\pi} + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{i2}^{(k)}(2\pi) \mu^k], \ldots | = 0, \end{aligned} \qquad (24)$$

according as the solutions for $\mu = 0$ are of the form (19) or (23), where the terms of the determinant not written are of the same form as those in (22). When for $\mu = 0$ the characteristic equation has a root of higher order of multiplicity, a corresponding discussion must be made. If for $\mu = 0$ the characteristic equation has several multiple roots, a corresponding discussion must be made for each associated group of solutions.

§ 5. *Solutions when* $\alpha_i^{(0)}$ *Are Distinct and* $\alpha_i^{(0)} - \alpha_j^{(0)} \not\equiv 0$ *mod* $\sqrt{-1}$.

The part of (22) independent of μ is

$$D_0 = |c_{ij}(e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1)| = |c_{ij}| \prod_{j=1}^{n} (e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1),$$

and the initial conditions have been taken so that $|c_{ij}| = 1$. If (22) were an identity in μ, its n solutions would be the n solutions of $D_0 = 0$, viz., $\alpha = \alpha_j^{(0)}$. In the general case in which it is not an identity in μ let

$$\alpha = \alpha_k^{(0)} + \beta_k. \tag{25}$$

Then we get

$$D = D_0 + \mu F_k(\beta_k, \mu) = (e^{-2\beta_k\pi} - 1) \prod_{j=1}^{n}{}' (e^{2(\alpha_j^{(0)}-\alpha_k^{(0)}-\beta_k)\pi} - 1) \\ + \mu F_k(\beta_k, \mu) = 0, \qquad j \neq k, \tag{26}$$

where $F_k(\beta_k, \mu)$ is a power series in μ and β_k, converging for

$$|\beta_k| < \infty, \qquad |\mu| < \rho > 0.$$

Since by hypothesis no $\alpha_j^{(0)} - \alpha_k^{(0)}$ equals an imaginary integer, the expansion of (26) as a power series in β_k and μ contains a term in β_k of the first degree and no term independent of both β_k and μ. Therefore by the theory of implicit functions (26) can be solved uniquely for β_k as a power series in μ of the form

$$\beta_k = \mu P_k(\mu), \tag{27}$$

which converges for $|\mu| > 0$ but sufficiently small. When we substitute this value of $\alpha = \alpha_k^{(0)} + \beta_k$ back in (21), we have n homogeneous linear equations among the A_j whose determinant is zero, but whose first minors, for $\mu = 0$, are not all zero; since, by hypothesis, for $\mu = 0$ the roots of the determinant set equal to zero are all distinct and no two differ by an imaginary integer. Therefore the ratios of the A_j are uniquely determined by these equations as power series in μ, converging for $|\mu|$ sufficiently small. Substituting the ratios

of the A_j in (20), we have the particular solution y_{ik}, $i=1, \ldots, n$, carrying one arbitrary constant and expanded as a power series in μ. Hence we may write it

$$y_{ik} = a_k \sum_{j=0}^{\infty} y_{ik}^{(j)} \mu^j. \tag{28}$$

Since the periodicity conditions have been satisfied,

$$y_{ik}(t+2\pi) - y_{ik}(t) = \sum_{j=0}^{\infty} [y_{ik}^{(j)}(t+2\pi) - y_{ik}^{(j)}(t)]\mu^j = 0$$

for every μ whose modulus is sufficiently small and for all t. Therefore

$$y_{ik}^{(j)}(t+2\pi) - y_{ik}^{(j)}(t) = 0, \qquad j = 0, \ldots, \infty,$$

from which it follows that each $y_{ik}^{(j)}$, $j = 0, \ldots, \infty$, is separately periodic. A solution is found similarly for each $\alpha_j^{(0)}$.

§ 6. *Solutions when no Two $\alpha_j^{(0)}$ Are Equal but when $\alpha_2^{(0)} - \alpha_1^{(0)} \equiv 0 \bmod \sqrt{-1}$*

Suppose two roots of the characteristic equation (18) for $\mu = 0$, say $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, differ by an imaginary integer and that there is no other such congruence among the $\alpha_j^{(0)}$. Then the equation corresponding to (26) becomes

$$D = (e^{-2\beta_1\pi} - 1)^2 \prod_{j=3}^{n} (e^{2(\alpha_j^{(0)} - \alpha_1^{(0)} - \beta_1)\pi} - 1) + \beta_1 \mu F_1(\beta_1, \mu) + \mu^2 F_2(\beta_1, \mu) = 0. \tag{29}$$

The term of lowest degree in β_1 alone is $+4\pi^2\beta_1^2$. The terms independent of β carry μ^2 as a factor, for every element of the first two columns of D, equation (24), in this case carries either β_1 or μ as a factor. In order to get the terms in μ alone we suppress those involving β_1, after which we get a factor μ from each of the first two columns. In general the term of lowest degree in μ alone will be in this case of the second degree.

In a similar manner if p of the $\alpha_j^{(0)}$ are congruent to zero mod $\sqrt{-1}$, then the term of lowest degree in β_1 alone is of degree p, and in μ alone it is at least of degree p.

The problem of the form of the solution of (29) is one of implicit functions. Writing the first terms explicitly, we have

$$\beta_1^2 + \gamma_{11}\beta_1\mu + \gamma_{02}\mu^2 + \text{terms of higher degree} = 0,$$

where $\gamma_{11}, \gamma_{02}, \ldots$ are constants independent of β_1 and μ. The quadratic terms may be factored and we get

$$(\beta_1 - b_1\mu)(\beta_1 - b_2\mu) + \text{terms of higher degree} = 0.$$

If b_1 and b_2 are distinct, the two solutions have the form

$$\left.\begin{aligned} \beta_{11} &= b_1\mu + \mu^2 P_1(\mu), \\ \beta_{12} &= b_2\mu + \mu^2 P_2(\mu), \end{aligned}\right\} \tag{30}$$

where P_1 and P_2 are power series in μ which converge if $|\mu|$ is sufficiently small. In this case the solutions are found precisely as in § 5.

If $b_1 = b_2$, the character of the solution depends upon terms of higher degree. It will proceed in powers of $\pm\sqrt{\mu}$ unless the $\theta_{ij}^{(1)}$ satisfy special conditions. But for special values of the $\theta_{ij}^{(1)}$ it may proceed according to integral powers of μ. We shall consider in detail the case where the series is in powers of $\pm\sqrt{\mu}$.

We see that the expansion of α_1 as a power series in $\sqrt{\mu}$ will contain no term in $\sqrt{\mu}$ to the first power, but will have the form

$$\alpha_1 = \alpha_1^{(0)} + 0\mu^{\frac{1}{2}} + \alpha_1^{(1)}\mu + \alpha_1^{(\frac{3}{2})}\mu^{\frac{3}{2}} + \dots .$$

Suppose this series has been obtained from equation (29). Then since not all of the first minors of D are zero the ratios of the A_j will be determined from (21). Suppose $\mu\delta$ is a non-vanishing first minor of D formed from the elements of its last $n-1$ columns. Then it follows from the form of (22) that, solving (21), we get

$$A_2 = \frac{\mu\delta_2}{\mu\delta}A_1, \quad A_j = \frac{\mu^2\delta_j}{\mu\delta}A_1, \qquad j = 3, \dots, n,$$

where

$$\begin{aligned} \delta &= \delta^{(0)} + \delta^{(\frac{1}{2})}\mu^{\frac{1}{2}} + \delta^{(1)}\mu + \dots, \\ \delta_j &= \delta_j^{(0)} + \delta_j^{(\frac{1}{2})}\mu^{\frac{1}{2}} + \delta_j^{(1)}\mu + \dots, \qquad j = 2, \dots, n. \end{aligned}$$

Substituting these series for the A_j in (20), we find that the y_{i1} are developable as series of the form

$$y_{i1} = y_{i1}^{(0)} + y_{i1}^{(\frac{1}{2})}\mu^{\frac{1}{2}} + y_{i1}^{(1)}\mu + \dots, \qquad i = 1, \dots, n.$$

Therefore the y_{i1} carry terms in $\mu^{\frac{1}{2}}$, although the term in $\sqrt{\mu}$ is absent in the expression for α_1.

If all the first minors obtained from D when the first column is suppressed are zero, and if there is a first minor which is distinct from zero when the second column is suppressed, the results are precisely the same. But suppose all the first minors obtained by omitting in turn the first and second columns are zero. Since the determinant has simple roots, there is at least one minor of the first order which is distinct from zero. Suppose it is obtained when the kth column

is suppressed. It follows from (22) that when $\alpha = \alpha_1$ it will carry the factor μ^2; let it be $\mu^2\delta$. Then solving (21), we get

$$A_1 = \frac{\mu\delta_1}{\mu^2\delta} A_k, \quad A_2 = \frac{\mu\delta_2}{\mu^2\delta} A_k, \quad A_j = \frac{\mu^2\delta_j}{\mu^2\delta} A_k, \qquad j = 3, \ldots, n,$$

where δ_1, δ_2, δ_j do not in general vanish for $\mu = 0$. It follows from the first two equations that A_k must carry μ as a factor. Hence in this case the y_{i1} have the same form as before. The y_{i2} have the same properties. These properties of the y_{i1} and the y_{i2} are necessary for the construction of the solution.

We now return to the consideration of (29). If the discriminant of the quadratic terms of (29),

$$D = \gamma_{11}^2 - 4\gamma_{20},$$

is distinct from zero, the solutions are in integral powers of μ, and at least one of them starts with a term of the first degree in μ. But if the discriminant is zero, then the character of the solutions depends upon the coefficients of terms of higher degree. They may be either in integral powers of μ or in powers of $\sqrt{\mu}$. If the solutions are in $\sqrt{\mu}$, they are real when μ has one sign and complex when it has the other. But if the solutions are in integral powers of μ, they are either real or complex for both positive and negative values of μ.

In all cases we get two solutions associated with the root $\alpha_1^{(0)}$. We should also get two solutions if we started from the root $\alpha_2^{(0)}$, but it follows from the form of (22) that they would not differ from those obtained by starting from $\alpha_1^{(0)}$. For each $\alpha_1^{(0)} + \beta_1$ the ratios of the A_j are determined from (21), and the results substituted in (20) give the y_i. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are found as in the preceding case. If there are several groups of $a^{(0)}$ in which these congruences exist, the discussion is made similarly for each one of them.

§ 7. *Solutions when $\alpha_1^{(0)}$ is a Multiple Root.*

Suppose that two and only two of the $\alpha_j^{(0)}$, viz. $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$, are equal, and that there are none of the congruences treated in § 6. Then for $\mu = 0$ we get from (22) and (24) either

$$D_0 = | c_{i1}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^0-\alpha)\pi} - 1), \; \ldots, \quad c_{ij}(e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1), \; \ldots |,$$

or

$$D_0 = | c_{i1}(e^{2(\alpha_1^0-\alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1) + 2\pi c_{i1} e^{2(\alpha_1^{(0)}-\alpha)\pi}, \; \ldots, \\ c_{ij}(e^{2(\alpha_j^{(0)}-\alpha)\pi} - 1), \; \ldots |,$$

both of which, by the theory of determinants, reduce to

$$D_0 = |c_{ij}| (e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1)^2 \prod_{j=3}^{n} (e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1), \; (|c_{ij}| = 1). \tag{31}$$

If we let $\alpha = \alpha_1^{(0)} + \beta_1$, as before, and expand as a power series in β_1, we find that the term of lowest degree in β_1 alone is $4\pi^2\beta_1^2$. When the determinant D is of the form (22) with $\alpha_2^{(0)} = \alpha_1^{(0)}$, the term of lowest degree in μ alone is at least of the second degree; but when D is of the form (24), which is the general case, the term of lowest degree in μ alone is in general of the first degree. Except in the special cases, the solutions for β_1 are therefore of the form

$$\left.\begin{aligned} \beta_{11} &= \mu^{\frac{1}{2}} P(\mu^{\frac{1}{2}}), \\ \beta_{12} &= -\mu^{\frac{1}{2}} P(-\mu^{\frac{1}{2}}), \end{aligned}\right\} \tag{32}$$

where P is a power series in $\mu^{\frac{1}{2}}$ containing a term independent of μ. The treatment of the special cases proceeds as in § 6. Substituting these expansions for $\alpha_1 = \alpha_1^{(0)} + \beta_1$ in (21) the ratios of the A_j are determined as power series in $\sqrt{\mu}$, and these results substituted in (21) give the y_{i1} and y_{i2} as power series in $\sqrt{\mu}$.

If, for $\mu = 0$, p roots, $\alpha_1^{(0)}, \ldots, \alpha_p^{(0)}$, are equal, then for these roots the expansion of D starts with β_1^p as the term of lowest degree in β_1 alone, and, except in special cases corresponding to those above mentioned when two roots are equal, the term of lowest degree in μ alone is of the first degree. Consequently in general for $\alpha_1^{(0)} = \ldots = \alpha_p^{(0)}$ we have

$$\beta_{1j} = \varepsilon^j \mu^{\frac{1}{p}} P(\varepsilon^j \mu^{\frac{1}{p}}), \qquad j = 0, \ldots, p-1,$$

where ε is a pth root of unity.

§ 8. *Solutions when there Are Equalities and Congruences among the Roots of the Characteristic Equation.*

Suppose, for example, that $\alpha_1^{(0)} = \alpha_2^{(0)}$ and $\alpha_3^{(0)}$ differs from $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$ by an imaginary integer, and that there are no other equalities or congruences among the $\alpha_j^{(0)}$. There are two cases, (a) where the solutions are of the form (19) with $\alpha_1^{(0)} = \alpha_2^{(0)}$, and (b) where the solutions are of the form (23).

Case (a). In this case $D_0 = 0$ becomes

$$D_0 = | c_{i1}(e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1), \quad c_{i2}(e^{2(\alpha_1^{(0)} - \alpha)\pi} - 1), \quad c_{i3}(e^{2(\alpha_3^{(0)} - \alpha)\pi} - 1), \ldots, \quad c_{ij}(e^{2(\alpha_j^{(0)} - \alpha)\pi} - 1), \ldots | = 0.$$

Putting $\alpha = \alpha_1^{(0)} + \beta_1$, we find that the term of lowest degree in β_1 is $-8\pi^3\beta_1^3$. In order to get the term of D of lowest degree in μ we put $\beta_1 = 0$. Then the first three columns of D are divisible by μ, while the others do not contain μ as factor. Consequently the term of lowest degree in μ is of the third degree at least. Moreover, since the first three columns of D vanish with $\beta_1 = \mu = 0$, there are no terms lower than the third degree in β_1 and μ. Hence in general D, in case (a), is of the form

$$D = \beta_1^3 + \gamma_{21}\beta_1^2\mu + \gamma_{12}\beta_1\mu^2 + \gamma_{03}\mu^3 + \ldots = 0. \tag{33}$$

Since the coefficients of this equation are real, it always has at least one real solution for β_1, vanishing with μ. The details of the various special cases are simply those of implicit functions. In general the three values of β_1 are expansible in integral powers of μ.

Case (b). In this case $D_0 = 0$ becomes

$$D_0 = | c_{i1}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1),\ c_{i2}(e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1) + 2\pi c_{i1}e^{2(\alpha_1^{(0)}-\alpha)\pi},\ c_{i3}(e^{2(\alpha_3^{(0)}-\alpha)\pi} - 1),\ \ldots\ | = 0.$$

By the theory of determinants this equation reduces to

$$D_0 = (e^{2(\alpha_1^{(0)}-\alpha)\pi} - 1)^2(e^{2(\alpha_3^{(0)}-\alpha)\pi} - 1)\prod_{j=4}^{n}(e^{2(\alpha_j^{(0)}-\alpha)} - 1) = 0.$$

Introducing β_1 as before, we find that the term of lowest degree in β_1 alone is $-8\pi^3\beta_1^3$. But when the terms involving μ are retained in D, the terms $2\pi c_{i1}e^{2(\alpha_1^{(0)}-\alpha)\pi}$ can not be eliminated from the second column. Hence only the first two columns vanish for $\beta_1 = \mu = 0$, and therefore in general in this case the expansion of D will contain a term in μ^2 alone. Since the first two columns vanish for $\beta_1 = \mu = 0$, there will be no terms of degree lower than the second in β_1 and μ. Hence in general D, in case (b), has the form

$$D = \beta_1^3 + \gamma_{11}\beta_1\mu + \gamma_{02}\mu^2 + \ldots = 0. \tag{34}$$

In the general case in which $\gamma_{11} \neq 0$ and $\gamma_{02} \neq 0$ there is one real solution in integral powers of μ and two solutions in $\sqrt{\mu}$. The two latter are real or imaginary for $\mu > 0$ according as γ_{11} is negative or positive. In all cases there is at least one real solution.

The treatment of cases where there is a higher order of multiplicity of the roots $\alpha_j^{(0)}$ and more numerous congruences is similar. If the total number of roots equal to $\alpha_1^{(0)}$ is ν_1, and of those congruent to $\alpha_1^{(0)}$ mod $\sqrt{-1}$ is ν_2, then in

the expansion of D the term of lowest degree in β_1 alone is of degree $\nu_1 + \nu_2$, and the term of lowest degree in μ alone is in general of degree ν_1. There are no terms of degree lower than ν_1 in β_1 and μ.

§ 9. *Solutions when $D = 0$ Has Two Roots Equal Identically in μ.*

The conditions necessary that $D = 0$ shall have two or more roots identical in μ are that

$$D(\alpha, \mu) = 0, \quad \frac{\partial}{\partial \alpha} D(\alpha, \mu) = 0$$

for all $|\mu|$ sufficiently small. Let us suppose, for simplicity, that α_1 and α_2 are identically equal in μ and that all the other α_j are distinct. In this case D_0 has the form (31), but there are no terms in D of the first degree in μ, for then β_{11} and β_{12} could not be equal. The value of $\beta_{11} = \beta_{12}$ is found as in § 7, and the corresponding solution is obtained by solving (21) for the ratios of the A_j and substituting the results in (20). If all the first minors of D vanish for $\alpha = \alpha_1$, then two of the A_j remain arbitrary and we obtain in this way the two solutions belonging to α_1.

Suppose not all the first minors of D are zero for $\alpha = \alpha_1$. Then there remains but one arbitrary in the solution of (21) for the ratios of the A_j. Substituting this result in (20) we get the solution $y_{i1}(\mu, t)$, from which we have by the general transformation

$$x_{i1} = e^{\alpha_1 t} y_{i1}, \tag{35}$$

where the y_{i1} are power series in μ whose coefficients are periodic in t with the period 2π.

By the general theory of § 2 we know that the other solution depending upon α_1 is

$$x_{i2} = e^{\alpha_1 t}(y_{i2} + t y_{i1}), \qquad i = 1, \ldots, n, \tag{36}$$

where the y_{i2} are periodic in t with the period 2π. Substituting these expressions in the differential equations (17) and making use of the fact that the $e^{\alpha_1 t} y_{i1}$ are a solution, we get

$$y'_{i2} + \alpha_1 y_{i2} - \sum_{j=1}^{n} [a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k] y_{j2} = - y_{i1}, \qquad i = 1, \ldots, n. \tag{37}$$

If the right members of these equations are put equal to zero, they become precisely of the form of the equations satisfied by y_{i1}. Consequently the only solution of these equations which is periodic with the period 2π is $y_{i2} = y_{i1}$ plus

such particular integrals that (37) shall be satisfied when the right members are retained. The part y_{i1} is useless since it belongs to the first solution, which contains an arbitrary factor. The method of finding the particular integrals will be taken up in § 16. It will there be shown that the y_{i2} are also power series in μ.

When $D=0$ has a multiple root of higher order of multiplicity for all $|\mu|$ sufficiently small, the $y_{i1}, y_{i2}, \ldots, i=1, \ldots, n$, are found in succession, the problem for $y_{i2}, y_{i3}, \ldots$ being that of linear equations with right members analogous to (37).

§ 10. *Direct Construction of the Solutions when the $\alpha_i^{(0)}$ Are Distinct and $\alpha_j^{(0)} - \alpha_i^{(0)} \not\equiv 0 \bmod \sqrt{-1}$, $i=1, \ldots, n$, $j=1, \ldots, n$.*

The methods given above lead to the constructions of the solutions, but the work is very laborious. However, the knowledge of their properties which we have obtained and their expansibility as power series in μ lead us to convenient methods for their construction.

Under the hypotheses of this article it has been shown that there are n distinct values of α, viz. $\alpha_1, \ldots, \alpha_n$, expansible as power series in μ such that

$$x_{ik} = e^{\alpha_k t} y_{ik}, \qquad i=1, \ldots, n,$$

constitute a fundamental set of solutions, where the y_{ik} are purely periodic and expansible as power series in μ. Since the y_{ik} are expansible as converging power series in μ and are periodic with the period 2π, each $y_{ik}^{(\lambda)}$ is separately periodic. Therefore the conditions on $\alpha = \alpha_j^{(0)} + \alpha_j^{(1)}\mu + \ldots$ and the $y_{ik}^{(\lambda)}$ are that they shall satisfy the differential equations identically in μ and

$$y_{ik}^{(\lambda)}(t+2\pi) - y_{ik}^{(\lambda)}(t) = 0. \tag{38}$$

The differential equations for the y_i are

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \Big[a_{ij} + \sum_{\lambda=1}^{\infty} \theta_{ij}^{(\lambda)} \mu^\lambda\Big] y_j, \qquad i=1, \ldots, n. \tag{39}$$

For $\mu=0$ the roots of the characteristic equation are $\alpha_1^{(0)}, \ldots, \alpha_n^{(0)}$. Consider any one of them, as $\alpha_k^{(0)}$. Then α and the y_i are expressible by converging power series of the form

$$\left.\begin{aligned} \alpha &= \alpha_k^{(0)} + \alpha_k^{(1)}\mu + \ldots = \sum_{\nu=0}^{\infty} \alpha_k^{(\nu)}\mu^\nu, \\ y_{ik} &= y_{ik}^{(0)} + y_{ik}^{(1)}\mu + \ldots = \sum_{\nu=0}^{\infty} y_{ik}^{(\nu)}\mu^\nu, \qquad i=1, \ldots, n. \end{aligned}\right\} \tag{40}$$

Substituting (40) in (39) and equating coefficients of corresponding powers of μ, we have a series of sets of differential equations from which α_k and the y_{ik} can be determined so that the y_{ik} shall be periodic with the period 2π. The determination is unique except for the arbitrary constant factor of the solution. For simplicity this arbitrary will be determined so that $y_{1k}(0) = c_{1k}$. If c_{1k} were zero, the initial condition would be imposed upon another $y_{ik}^{(0)}$, not all of which can vanish at $t = 0$. The arbitrary can be restored if desired in the final solution by multiplying by any factor not zero. We shall consider only enough steps of the process to enable us to exhibit its peculiarities and to establish its general character.

Terms Independent of μ. The terms of the solution independent of μ are defined by the differential equations

$$(y_{ik}^{(0)})' + \alpha_k^{(0)} y_{ik}^{(0)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(0)} = 0, \qquad i = 1, \ldots, n,$$

the general solution of which is

$$y_{ik}^{(0)} = \sum_{j=1}^{n} \eta_{jk}^{(0)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_k^{(0)})t}, \qquad i = 1, \ldots, n,$$

where the $\eta_{jk}^{(0)}$ are the constants of integration. Since the $y_{ik}^{(0)}$ are to be made periodic with the period 2π, and since $\alpha_j^{(0)} - \alpha_k^{(0)} \not\equiv 0 \bmod \sqrt{-1}$ by hypothesis except when $j = k$, it follows that $\eta_{jk}^{(0)} = 0$ when $j \neq k$. Since $y_{1k}(0) = c_{1k}$ for all $|\mu|$ sufficiently small, it follows that $y_{1k}^{(0)}(0) = c_{1k}$, $y_{1k}^{(j)}(0) = 0, j = 1, \ldots, \infty$. Therefore we must make $\eta_{kk} = 1$. The solution satisfying the conditions laid down is therefore found to be

$$y_{ik}^{(0)} = c_{ik}, \qquad i = 1, \ldots, n. \tag{41}$$

Coefficients of μ. The differential equations for the coefficients of the first power of μ are

$$(y_{ik}^{(1)})' + \alpha_k^{(0)} y_{ik}^{(1)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(1)} = -\alpha_k^{(1)} y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{jk}^{(0)}, \qquad i = 1, \ldots, n. \tag{42}$$

When the left members of these equations are set equal to zero their general solution is

$$y_{ik}^{(1)} = \sum_{j=1}^{n} \eta_{jk}^{(1)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_k^{(0)})t}, \qquad i = 1, \ldots, n, \tag{43}$$

where the $\eta_{jk}^{(1)}$ are the constants of integration, and the c_{ij} are the same as before.

We shall find the complete solutions of (42) by the method of the variation of parameters. Regarding the $\eta_{jk}^{(1)}$ as variables and imposing the conditions that (43) shall satisfy (42), we get

$$\sum_{j=1}^{n}(\eta_{jk}^{(1)})' c_{ij} e^{(\alpha_j^{(0)}-\alpha_k^{(0)})t} = -\alpha_k^{(1)} y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{jk}^{(0)} = g_{ik}^{(1)}(t), \qquad i = 1, \ldots, n, \quad (44)$$

where the $g_{ik}^{(1)}(t)$ are periodic in t with the period 2π. The determinant of the coefficients of the $(\eta_{jk}^{(1)})'$ is

$$\Delta = |c_{ij}| e^{\sum_{j=1}^{n}(\alpha_j^{(0)}-\alpha_k^{(0)})t} = e^{\sum_{j=1}^{n}(\alpha_j^{(0)}-\alpha_k^{(0)})t},$$

which can not vanish for any finite t. Therefore the solutions of (44) for the $(\eta_{jk}^{(1)})'$ are

$$(\eta_{jk}^{(1)})' = e^{-(\alpha_j^{(0)}-\alpha_k^{(0)})t} \Delta_{jk}^{(1)}, \quad (45)$$

where the $\Delta_{jk}^{(1)}$ are known periodic functions of t with the period 2π.

For $j \neq k$ the solutions of (45) have the form

$$\eta_{jk}^{(1)} = e^{-(\alpha_j^{(0)}-\alpha_k^{(0)})t} P_{jk}(t) + B_{jk}^{(1)}, \quad (46)$$

where the $P_{jk}^{(1)}(t)$ are periodic with the period 2π and the $B_{jk}^{(1)}$ are arbitrary constants. For $j = k$ we have, from (44),

$$(\eta_{kk}^{(1)})' = \Delta_{kk}^{(1)} = -\alpha_k^{(1)} + \delta_{kk}^{(1)}, \quad (47)$$

where $\delta_{kk}^{(1)}$ is $\Delta_{kk}^{(1)}$ after the terms $-\alpha_k^{(1)} y_{ik}^{(0)}$ have been omitted from the kth column. It is a periodic function of t with the period 2π and has in general a term independent of t. Hence we may write

$$\delta_{kk}^{(1)} = d_k^{(1)} + Q_k^{(1)}(t),$$

where $d_k^{(1)}$ is constant and the mean value of $Q_k^{(1)}(t)$ is zero. Then (47) becomes

$$(\eta_{kk}^{(1)})' = (d_k^{(1)} - \alpha_k^{(1)}) + Q_k^{(1)}(t). \quad (48)$$

In order that $\eta_{kk}^{(1)}$ shall be periodic we must impose the condition

$$\alpha_k^{(1)} = d_k^{(1)}, \quad (49)$$

after which we get

$$\eta_{kk}^{(1)} = P_{kk}^{(1)} + B_{kk}^{(1)}, \quad (50)$$

where $P_{kk}^{(1)}$ is periodic with the period 2π and $B_{kk}^{(1)}$ is the constant of integration.

Substituting (46) and (50) in (43), we have as the general solution of (42)

$$y_{ik}^{(1)} = \sum_{j=1}^{n} B_{jk}^{(1)} c_{ij} e^{(\alpha_j^{(0)}-\alpha_k^{(0)})t} + \sum_{j=1}^{n} c_{ij} P_{jk}^{(1)}(t), \qquad i = 1, \ldots, n. \quad (51)$$

In order that the $y_{ik}^{(1)}$ shall be periodic with the period 2π, all the $B_{jk}^{(1)}$ must be zero except $B_{kk}^{(1)}$. From $y_{1k}^{(1)}(0)=0$, we get

$$B_{kk}^{(1)} = -\frac{1}{c_{1k}}\sum_{j=1}^{n} c_{1j} P_{jk}^{(1)}(0). \tag{52}$$

Therefore the solution satisfying all the conditions is

$$y_{ik}^{(1)} = \sum_{j=1}^{n}\left[c_{ij} P_{jk}^{(1)}(t) - \frac{c_{ik}}{c_{1k}} c_{1j} P_{jk}^{(1)}(0)\right]. \tag{53}$$

It remains to be shown that the integration of the coefficients of the higher powers of μ can be effected in a similar manner. Suppose $\alpha_k^{(1)}, \ldots, \alpha_k^{(m-1)}$ and the $y_{ik}^{(1)}, y_{ik}^{(2)}, \ldots, y_{ik}^{(m-1)}$ satisfying the differential equations have been uniquely determined so that the $y_{ik}^{(l)}$ are periodic with the period 2π and that $y_{1k}^{(l)}(0)=0$, $l=1, \ldots, m-1$. We shall show that the $y_{ik}^{(m)}$ can be determined so as to satisfy the same conditions.

From equations (39) and (40) we find

$$\begin{aligned}(y_{ik}^{(m)})' + \alpha_k^{(0)} y_{ik}^{(m)} - \sum_{j=1}^{n} a_{ij} y_{jk}^{(m)} = -\alpha_k^{m} y_{ik}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(m)} y_{jk}^{(0)} \\ + \sum_{p=1}^{m-1}\Big[-\alpha_k^{(p)} y_{ik}^{(m-p)} + \sum_{j=1}^{n} \theta_{ij}^{(p)} y_{jk}^{(m-p)}\Big].\end{aligned} \tag{54}$$

Omitting the terms included under the sign of summation with respect to p, these equations are identical in form with (42) except that we now have the superscript (m) instead of (1). The integrations proceed as in the case treated, for the terms included under the summation with respect to p are all periodic with the period 2π and do not change the character of the $g_{ik}^{(m)}(t)$. Therefore $\alpha_k^{(m)}$ and the $y_{ik}^{(m)}$ can be uniquely determined so as to satisfy the differential equations and be periodic in t with the period 2π, and so that at the same time $y_{1k}^{(m)}(0)=0$. Hence the induction is complete and the process can be indefinitely continued.

§ 11. *Construction of the Solutions when $\alpha_2^{(0)} - \alpha_1^{(0)} \equiv 0 \mod \sqrt{-1}$.*

Suppose the $\alpha_j^{(0)}$ are all distinct, that $\alpha_2^{(0)}$ and $\alpha_1^{(0)}$ differ by an imaginary integer, and that there are no other such congruences among the $\alpha_j^{(0)}$. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are computed by the methods of § 10. It was shown in § 6 that in this case, in general, α_1, α_2 and the y_{i1}, y_{i2} can be developed as converging series in integral powers of μ. It will be assumed

that we are not treating one of the exceptional cases where the series proceed in fractional powers of μ.

The general solution of (39) for the terms independent of μ is in this case

$$y_{i1}^{(0)} = \sum_{j=1}^{n} \eta_{j1}^{(0)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t}, \qquad i = 1, \ldots, n,$$

where the $\eta_{j1}^{(0)}$ are the constants of integration.

Imposing the conditions that the $y_{i1}^{(0)}$ shall be periodic with the period 2π and that $y_{11}^{(0)}(0) = c_{11}$, these equations become, since $\alpha_2^{(0)} - \alpha_1^{(0)}$ is an imaginary integer,

$$y_{i1}^{(0)} = \left(1 - \eta_{21}^{(0)} \frac{c_{12}}{c_{11}}\right) c_{i1} + \eta_{21}^{(0)} c_{i2} e^{(\alpha_2^{(0)} - \alpha_1^{(0)})t}, \qquad i = 1, \ldots, n, \tag{55}$$

where $\eta_{21}^{(0)}$ is so far arbitrary.

Coefficients of μ. It follows from (39) and (40) that the coefficients of μ must satisfy the differential equations

$$(y_{i1}^{(1)})' + \alpha_1^{(0)} y_{i1}^{(1)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(1)} = -\alpha_1^{(1)} y_{i1}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}, \qquad i = 1, \ldots, n. \tag{56}$$

The general solution of these equations when their right members are zero is

$$y_{i1}^{(1)} = \sum_{j=1}^{n} \eta_{j1}^{(1)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t}, \qquad i = 1, \ldots, n. \tag{57}$$

Considering the coefficients $\eta_{j1}^{(1)}$ as functions of t and imposing the conditions that (56) shall be satisfied, we get

$$\sum_{j=1}^{n} (\eta_{j1}^{(1)})' c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)})t} = -\alpha_1^{(1)} y_{i1}^{(0)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}, \qquad i = 1, \ldots, n.$$

Substituting the values of $y_{i1}^{(0)}$ from (55) and solving, it is found that

$$\left.\begin{aligned} (\eta_{11}^{(1)})' &= -\alpha_1^{(1)}\left(1 - \eta_{21}^{(0)} \frac{c_{12}}{c_{11}}\right) + \eta_{21}^{(0)} \Delta_{11}^{(1)}(t) + D_{11}^{(1)}(t), \\ (\eta_{21}^{(1)})' &= -\alpha_1^{(1)} \eta_{21}^{(0)} + \eta_{21}^{(0)} \Delta_{21}^{(1)}(t) + D_{21}^{(1)}(t), \\ (\eta_{j1}^{(1)})' &= e^{-(\alpha_j^{(0)} - \alpha_1^{(0)})t} \Delta_{j1}^{(1)}(t), \qquad j = 3, \ldots, n, \end{aligned}\right\} \tag{58}$$

where the $\Delta_{j1}^{(1)}$ and $D_{j1}^{(1)}$ are periodic functions of t with the period 2π depending upon the $\theta_{ij}^{(1)}$ and $e^{(\alpha_2^{(0)} - \alpha_1^{(0)})t}$. In the first two equations the undetermined constants $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ enter only as they are exhibited explicitly.

Equations (58) are to be integrated and the results substituted in (57). In order that the $y_{i1}^{(1)}$ shall be periodic we must impose the conditions

$$\left.\begin{aligned} -\alpha_1^{(1)}\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)+\eta_{21}^{(0)}b_{11}^{(1)}+d_{11}^{(1)}&=0,\\ -\alpha_1^{(1)}\eta_{21}^{(0)}+\eta_{21}^{(0)}b_{21}^{(1)}+d_{21}^{(1)}&=0,\\ B_{j1}^{(1)}&=0, \qquad j=3,\ldots,n, \end{aligned}\right\} \tag{59}$$

where $b_{11}^{(0)}$, $b_{21}^{(1)}$, $d_{11}^{(1)}$, $d_{21}^{(1)}$ are the constant terms of $\Delta_{11}^{(1)}$, $\Delta_{21}^{(1)}$, $D_{11}^{(1)}$ and $D_{21}^{(1)}$ respectively, and where the $B_{j1}^{(1)}$ are the constants of integration obtained with the last $n-2$ equations. Eliminating $\eta_{21}^{(0)}$ from the first two equations, we get

$$\alpha_1^{(1)^2}-\left(b_{21}^{(1)}+d_{11}^{(1)}+\frac{c_{12}}{c_{11}}d_{21}^{(1)}\right)\alpha_1^{(1)}+\left(b_{21}^{(1)}d_{11}^{(1)}-b_{11}^{(1)}d_{21}^{(1)}\right)=0. \tag{60}$$

There are two cases, according as the discriminant of this quadratic is not zero or is zero. In the first case, which may be regarded as the general case, the two roots for $\alpha_1^{(1)}$ are distinct, corresponding to distinct values of b_1 and b_2 given in (30) in the existence proof. It was shown there that in this case the solutions proceed according to integral powers of μ. In the second case, corresponding to $b_1=b_2$, the character of the solutions depends upon the coefficients of terms of higher degree, and they may proceed according to powers of μ or $\pm\sqrt{\mu}$. We shall assume that the discriminant is distinct from zero and proceed to the construction of the solutions.

It will be shown that after one of the two pairs of values of $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ satisfying (59) is chosen, the solution is uniquely determined except for the arbitrary constant factor which may be introduced at the end. Integrating (58), substituting the results in (57), and determining the arbitrary constants so that the solution shall be periodic, and imposing the condition that $y_{11}^{(1)}(0)=0$, we find

$$y_{i1}^{(1)}=B_{21}^{(1)}\left[-\frac{c_{12}}{c_{11}}c_{i1}+c_{i2}e^{(\alpha_2^{(0)}-\alpha_1^{(0)})t}\right]+\sum_{j=1}^{n}\left[c_{ij}P_{j1}^{(1)}(t)-\frac{c_{1j}}{c_{11}}c_{i1}P_{j1}^{(1)}(0)\right], \tag{61}$$
$$i=1,\ldots,n,$$

where $B_{21}^{(1)}$ is an undetermined constant and the $P_{j1}^{(1)}$ are entirely known periodic functions of t having the period 2π.

Coefficients of μ^2. The coefficients of μ^2 are defined by

$$(y_{i1}^{(2)})'+\alpha_1^{(0)}y_{i1}^{(2)}-\sum_{j=1}^{n}a_{ij}y_{j1}^{(2)}=-\alpha_1^{(2)}y_{i1}^{(0)}-\alpha_1^{(1)}y_{i1}^{(1)}+\sum_{j=1}^{n}\left[\theta_{ij}^{(2)}y_{j1}^{(0)}+\theta_{ij}^{(1)}y_{j1}^{(1)}\right], \tag{62}$$
$$i=1,\ldots,n.$$

The general solutions of these equations when the right members are put equal to zero are the same as (57) except that the superscripts are (2) instead of (1).

Varying the $\eta_{j1}^{(2)}$, we find for the equations corresponding to (58)

$$\left.\begin{aligned}
(\eta_{11}^{(2)})' &= -\alpha_1^{(2)}\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)+\alpha_1^{(1)}B_{21}^{(1)}\frac{c_{12}}{c_{11}}+B_{21}^{(1)}\Delta_{11}^{(1)}(t)+D_{11}^{(2)}(t),\\
(\eta_{21}^{(2)})' &= -\alpha_1^{(2)}\eta_{21}^{(0)}+\alpha_1^{(1)}B_{21}^{(1)}+B_{21}^{(1)}\Delta_{21}^{(1)}(t)+D_{21}^{(2)}(t),\\
(\eta_{j1}^{(2)})' &= e^{-(\alpha_j^{(0)}-\alpha_1^{(0)})t}\Delta_{j1}^{(2)}(t), \qquad j=3,\ \ldots,\ n.
\end{aligned}\right\} \tag{63}$$

The undetermined constants $\alpha_1^{(2)}$ and $B_{21}^{(1)}$ are written explicitly in the first two equations, and it is to be noted that $\Delta_{11}^{(1)}(t)$ and $\Delta_{21}^{(1)}(t)$ are precisely the same functions of t as those which appear in (58).

In order that when we integrate (63) and substitute the results in the equations corresponding to (57) we shall have a periodic solution, we must impose the conditions

$$\left.\begin{aligned}
-\left(1-\eta_{21}^{(0)}\frac{c_{12}}{c_{11}}\right)\alpha_1^{(2)}+\left(b_{11}^{(1)}+\alpha_1^{(1)}\frac{c_{12}}{c_{11}}\right)B_{21}^{(1)}+d_{11}^{(2)}&=0,\\
-\eta_{21}^{(0)}\alpha_1^{(2)}+(b_{21}^{(1)}-\alpha_1^{(1)})B_{21}^{(1)}+d_{21}^{(2)}&=0,\\
B_{j1}^{(2)}=0, \quad j=3,\ \ldots,\ n,&
\end{aligned}\right\} \tag{64}$$

$b_{11}^{(1)}$, $b_{21}^{(1)}$, $d_{11}^{(2)}$, $d_{21}^{(2)}$ being the constant terms of $\Delta_{11}^{(1)}$, $\Delta_{21}^{(1)}$, $D_{11}^{(2)}$ and $D_{21}^{(2)}$ respectively. The first two equations are linear in $\alpha_1^{(2)}$ and $B_{21}^{(1)}$ and determine these quantities uniquely, provided their determinant is not zero. The determinant is

$$\Delta=\begin{vmatrix} -1+\eta_{21}^{(0)}\dfrac{c_{12}}{c_{11}}, & b_{11}^{(1)}+\alpha_1^{(1)}\dfrac{c_{12}}{c_{11}} \\ -\eta_{21}^{(0)}, & b_{21}^{(1)}-\alpha_1^{(1)}\end{vmatrix}=\alpha_1^{(1)}+\eta_{21}^{(0)}\left(b_{11}^{(1)}+b_{21}^{(1)}\frac{c_{12}}{c_{11}}\right)-b_{21}^{(1)}.$$

Eliminating $\alpha_1^{(1)}$ and $\eta_{21}^{(0)}$ by means of (60) and (59), we get

$$\Delta=\pm\sqrt{D}, \tag{65}$$

where D is the discriminant of (60) and is by hypothesis distinct from zero. Therefore the solution of (64) for $\alpha_1^{(2)}$ and $B_{21}^{(1)}$ is unique. The sign before $\sqrt{D}$ depends upon which of the two roots of (60) is used. Now integrating (63), substituting the results in the equations corresponding to (57), and imposing the condition that $y_{11}^{(2)}(0)=0$, it follows that

$$y_{i1}^{(2)}=B_{21}^{(2)}\left[-\frac{c_{12}}{c_{11}}c_{i1}+c_{i2}e^{(\alpha_2^{(0)}-\alpha_1^{(0)})t}\right]+\sum_{j=1}^{n}\left[c_{ij}P_{j1}^{(2)}(t)-\frac{c_{1j}}{c_{11}}c_{i1}P_{j1}^{(2)}(0)\right], \tag{66}$$
$$i=1,\ \ldots,\ n,$$

where $B_{21}^{(2)}$ is as yet an undetermined constant.

Now consider the general step in the construction of the solution. We shall have

$$(y_{i1}^{(\nu)})' + \alpha_1^{(0)} y_{i1}^{(\nu)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(\nu)} = -\alpha_1^{(\nu)} y_{i1}^{(0)} - \alpha_1^{(1)} y_{i1}^{(\nu-1)} + F_i^{(\nu)}, \qquad i = 1, \ldots, n,$$

where the $F_i^{(\nu)}$ are known periodic functions of t. When we put the right members equal to zero, the general solutions are the same as (57) except that the superscripts are (ν). By varying the constants of integration, we get equations (63) except that the superscripts are (ν) and ($\nu - 1$) instead of (2) and (1) respectively. The conditions for periodicity are similar to (64) and have the same determinant. Consequently at this step $\alpha_1^{(\nu)}$ and $B_{21}^{(\nu-1)}$ are uniquely determined. Hence it is evident that the process can be carried as far as is desired.

For congruences of higher order analogous methods are applicable, and in the exceptional cases to this treatment the existence proof furnishes a sure guide for the construction of the solutions.

§ 12. *Construction of the Solutions when* $\alpha_2^{(0)} = \alpha_1^{(0)}$.

For simplicity, suppose $\alpha_2^{(0)} = \alpha_1^{(0)}$ and that there are no other equalities among the $\alpha_j^{(0)}$, and that no two of them differ by an imaginary integer. The only variations from the method of § 10 are in the construction of the solutions associated with $\alpha_1^{(0)}$. It was shown in § 7 that, except in special cases, the solutions are in powers of $\pm\sqrt{\mu}$. We shall assume that we are not treating one of the special cases. Hence we have

$$\left.\begin{aligned} \alpha_1 &= \alpha_1^{(0)} + \alpha_1^{(1)} \mu^{\frac{1}{2}} + \alpha_1^{(2)} \mu + \cdots, \\ \alpha_2 &= \alpha_1^{(0)} - \alpha_1^{(1)} \mu^{\frac{1}{2}} + \alpha_1^{(2)} \mu - \cdots, \\ y_{i1} &= y_{i1}^{(0)} + y_{i1}^{(1)} \mu^{\frac{1}{2}} + y_{i1}^{(2)} \mu + \cdots, \\ y_{i2} &= y_{i1}^{(0)} - y_{i1}^{(1)} \mu^{\frac{1}{2}} + y_{i1}^{(2)} \mu - \cdots. \end{aligned}\right\} \qquad (67)$$

Terms Independent of μ. The terms independent of μ are defined by

$$(y_{i1}^{(0)})' + \alpha_1^{(0)} y_{i1}^{(0)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(0)} = 0, \qquad i = 1, \ldots, n.$$

The general solution of these equations is

$$y_{i1}^{(0)} = \eta_{i1}^{(0)} c_{i1} + \eta_{21}^{(0)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} \eta_{j1}^{(0)} c_{ij} e^{(\alpha_j^{(0)} - \alpha_1^{(0)}) t}.$$

Imposing the conditions that the $y_{i1}^{(0)}$ shall be periodic with the period 2π and that $y_{11}^{(0)}(0) = c_{11}$, we get

$$y_{i1}^{(0)} = c_{i1}, \qquad i = 1, \ldots, n. \qquad (68)$$

Coefficients of $\mu^{\frac{1}{2}}$. The coefficients of $\mu^{\frac{1}{2}}$ are defined by

$$(y_{i1}^{(1)})' + \alpha_1^{(0)} y_{i1}^{(1)} - \sum_{j=1}^{n} a_{ij}\, y_{j1}^{(1)} = -\alpha_1^{(1)} y_{i1}^{(0)} = -\alpha_1^{(1)} c_{i1}, \qquad i = 1, \ldots, n. \tag{69}$$

The general solution of these equations when the right members are zero is

$$y_{i1}^{(1)} = \eta_{11}^{(1)} c_{i1} + \eta_{21}^{(1)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} \eta_{j1}^{(1)} c_{ij}\, e^{(\alpha_j^{(0)} - \alpha_1^{(0)}) t}, \qquad i = 1, \ldots, n. \tag{70}$$

By the variation of parameters we get

$$(\eta_{11}^{(1)})' c_{i1} + (\eta_{21}^{(1)})' (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} (\eta_{j1}^{(1)})' c_{ij}\, e^{(\alpha_j^{(0)} - \alpha_1^{(0)}) t} = -\alpha_1^{(1)} c_{i1}, \quad i = 1, \ldots, n.$$

Solving these equations for the $(\eta_{j1}^{(1)})'$, we find

$$(\eta_{11}^{(1)})' = -\alpha_1^{(1)},$$
$$(\eta_{j1}^{(1)})' = 0, \qquad j = 2, \ldots, n.$$

Consequently

$$\left.\begin{aligned} \eta_{11}^{(1)} &= -\alpha_1^{(1)} t + B_{11}^{(1)}, \\ \eta_{j1}^{(1)} &= B_{j1}^{(1)}, \qquad j = 2, \ldots, n. \end{aligned}\right\} \tag{71}$$

Substituting these values of $\eta_{j1}^{(1)}$ in (70), we get

$$y_{i1}^{(1)} = [B_{11}^{(1)} - \alpha_1^{(1)} t]\, c_{i1} + B_{21}^{(1)} (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} B_{j1}^{(1)} c_{ij}\, e^{(\alpha_j^{(0)} - \alpha_1^{(0)}) t}, \ i = 1, \ldots, n. \tag{72}$$

Imposing the conditions that the $y_{i1}^{(1)}$ shall be periodic with the period 2π and that $y_{11}^{(1)}(0) = 0$, we have

$$\left.\begin{aligned} B_{21}^{(1)} &= \alpha_1^{(1)}, \\ B_{j1}^{(1)} &= 0, \qquad j = 3, \ldots, n, \\ B_{11}^{(1)} c_{11} &+ B_{21}^{(1)} c_{12} = 0. \end{aligned}\right\} \tag{73}$$

Then equations (72) become

$$y_{i1}^{(1)} = \left(-\frac{c_{12}}{c_{11}} c_{i1} + c_{i2}\right) \alpha_1^{(1)}, \qquad i = 1, \ldots, n, \tag{74}$$

where $\alpha_1^{(1)}$ remains as yet undetermined.

Coefficients of μ. The coefficients of μ satisfy

$$(y_{i1}^{(2)})' + \alpha_1^{(0)} y_{i1}^{(2)} - \sum_{j=1}^{n} a_{ij}\, y_{j1} = -\alpha_1^{(2)} y_{i1}^{(0)} - \alpha_1^{(1)} y_{i1}^{(1)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}, \quad i = 1, \ldots, n. \tag{75}$$

The solution of these equations when the right members are zero is of the same form as (70), and we find, by varying the constants,

$$(\eta_{11}^{(2)})' c_{i1} + (\eta_{21}^{(2)})' (c_{i2} + t c_{i1}) + \sum_{j=3}^{n} (\eta_{j1}^{(2)})' c_{ij}\, e^{(\alpha_j^{(0)} - \alpha_1^{(0)}) t} = -\alpha_1^{(2)} y_{i1}^{(0)} - \alpha_1^{(1)} y_{i1}^{(1)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(0)}.$$

Solving these equations, we get

$$\left.\begin{aligned}(\eta_{11}^{(2)})' &= -\alpha_1^{(2)} + \left(\frac{c_{12}}{c_{11}} + t\right)\alpha_1^{(1)^2} + t\,\Delta_{11}^{(2)}(t) + D_{11}^{(2)}(t),\\ (\eta_{21}^{(2)})' &= \qquad\qquad -\alpha_1^{(1)^2} - \Delta_{11}^{(2)}(t),\\ (\eta_{j1}^{(2)})' &= e^{-(\alpha_j^{(0)}-\alpha_1^{(0)})t}\,\Delta_{j1}^{(2)}(t), \qquad j = 3, \ldots, n,\end{aligned}\right\} \quad (76)$$

where $\Delta_{j1}^{(2)}(t)$ and $D_{11}^{(2)}(t)$ are periodic functions of t. The first of these equations gives rise to integrals of the type

$$a_j \int t \begin{matrix}\sin\\ \cos\end{matrix} jt\,dt = \mp \frac{a_j}{j}\, t \begin{matrix}\cos\\ \sin\end{matrix} jt + \frac{a_j}{j^2}\begin{matrix}\sin\\ \cos\end{matrix} jt.$$

The second equation gives rise to the corresponding integral

$$-a_j \int \begin{matrix}\sin\\ \cos\end{matrix} jt\,dt = \pm \frac{a_j}{j}\begin{matrix}\cos\\ \sin\end{matrix} jt.$$

When we substitute these results in the equations corresponding to (70), we get for these terms

$$(\eta_{11}^{(2)} + \eta_{21}^{(2)} t)\,c_{i1} = \frac{a_j}{j}\left[\mp t \begin{matrix}\cos\\ \sin\end{matrix} jt \pm t \begin{matrix}\cos\\ \sin\end{matrix} jt\right] + \frac{a_j}{j^2}\begin{matrix}\sin\\ \cos\end{matrix} jt.$$

Consequently the terms of the type $t \begin{matrix}\cos\\ \sin\end{matrix} jt$ vanish when the results of the integration of (76) are substituted back in the equations corresponding to (70). Hence, we get at this step

$$\left.\begin{aligned}y_{i1}^{(2)} = B_{11}^{(2)} c_{i1} &+ B_{21}^{(2)}(c_{i2} + t\,c_{i1}) + \sum_{j=3}^{n} B_{j1}^{(2)} c_{ij}\, e^{(\alpha_j^{(0)}-\alpha_1^{(0)})t}\\ &+ \left[\left(-\alpha_1^{(2)} + \frac{c_{12}}{c_{11}}\alpha_1^{(1)^2} + d_{11}^{(2)}\right)t + \tfrac{1}{2}(\alpha_1^{(1)^2} + b_{11}^{(2)})\,t^2 + P_{11}^{(2)}(t)\right]c_{i1}\\ &+ \left[-(\alpha_1^{(1)^2} + b_{11}^{(2)})\,t + P_{21}^{(2)}(t)\right]c_{i2} + \sum_{j=3}^{n} c_{ij} P_{j1}^{(2)}(t),\end{aligned}\right\} \quad (77)$$

where $b_{11}^{(2)}$ and $d_{11}^{(2)}$ are the constant terms in $\Delta_{11}^{(2)}$ and $D_{11}^{(2)}$, and where the $P_{j1}^{(2)}(t)$ are periodic functions of t. In order that the $y_{i1}^{(2)}$ shall be periodic and $y_{11}^{(2)}(0) = 0$, we must impose the conditions

$$\left.\begin{aligned}\alpha_1^{(1)} &= \pm\sqrt{-b_{11}^{(2)}},\\ B_{21}^{(2)} &= \alpha_1^{(2)} - \frac{c_{12}}{c_{11}} b_{11}^{(2)} - d_{11}^{(2)},\\ B_{j1}^{(2)} &= 0, \qquad j = 3, \ldots, n,\\ B_{11}^{(2)} c_{11} &+ B_{21}^{(2)} c_{12} + \sum_{j=1}^{n} P_{j1}^{(2)}(0)\, c_{1j} = 0.\end{aligned}\right\} \quad (78)$$

The only undetermined constant remaining in (77) is $\alpha_1^{(2)}$, and the $y_{i1}^{(2)}$ now have the form

$$y_{i1}^{(2)} = \left(c_{i2} - \frac{c_{12}}{c_{11}} c_{i1}\right) \alpha_1^{(2)} + \Phi_{i1}^{(2)}(t), \qquad i = 1, \ldots, n, \tag{79}$$

where the $\Phi_{i1}^{(2)}(t)$ are known periodic functions of t.

The constant $\alpha_1^{(1)}$ has a double determination except in the special case where $b_{11}^{(2)} = 0$. We shall suppose $b_{11}^{(2)} \neq 0$, for if it is zero we have one of the special cases excluded above. It will be shown that the work becomes unique after one of the two possible values of $\alpha_1^{(1)}$ is chosen.

The coefficients of $\mu^{\frac{3}{2}}$ are determined by

$$(y_{i1}^{(3)})' + \alpha_1^{(0)} y_{i1}^{(3)} - \sum_{j=1}^{n} a_{ij} y_{j1}^{(3)} = -\alpha_1^{(3)} y_{i1}^{(0)} - \alpha_1^{(2)} y_{i1}^{(1)} - \alpha_1^{(1)} y_{i1}^{(2)} + \sum_{j=1}^{n} \theta_{ij}^{(1)} y_{j1}^{(1)}.$$

The equations corresponding to (76) are

$$\begin{aligned}
(\eta_{11}^{(3)})' &= -\alpha_1^{(3)} + 2\left(\frac{c_{12}}{c_{11}} + t\right) \alpha_1^{(1)} \alpha_1^{(2)} + t\,\Delta_{11}^{(3)}(t) + D_{11}^{(3)}(t), \\
(\eta_{21}^{(3)})' &= \qquad\qquad - 2\,\alpha_1^{(1)} \alpha_1^{(2)} - \Delta_{11}^{(3)}(t), \\
(\eta_{j1}^{(3)})' &= e^{-(\alpha_j^{(0)} - \alpha_1^{(0)})t} \Delta_{j1}^{(3)}, \qquad j = 3, \ldots, n.
\end{aligned}$$

In order that the final solutions at this step shall be periodic, we must impose the condition

$$2\,\alpha_1^{(1)} \alpha_1^{(2)} + b_{11}^{(3)} = 0,$$

which determines $\alpha_1^{(2)}$ uniquely since $\alpha_1^{(1)} \neq 0$. The other constants are all uniquely determined by the periodicity condition and the initial condition except $\alpha_1^{(3)}$, which is fixed by the periodicity condition at the next step. Similarly, another solution is obtained using the other determination of $\alpha_1^{(1)}$. The solutions associated with $\alpha_3^{(0)}, \ldots, \alpha_n^{(0)}$ are obtained by the method of § 10.

The chief types of cases have been treated, and the exceptions to them are developed similarly, according to the forms indicated by the existence proofs.

§ 13. *Solutions when the θ_{ij} do not All Reduce to Constants for $\mu = 0$.*

Heretofore we have considered linear differential equations whose coefficients become constants when $\mu = 0$. We shall now waive this restriction, but we shall suppose the solutions are known for $\mu = 0$. Let the equations under consideration be

$$x_i' = \sum_{j=1}^{n} \left[\sum_{k=0}^{\infty} \theta_{ij}^{(k)}(t) \mu^k \right] x_j, \qquad i = 1, \ldots, n, \tag{80}$$

where the $\theta_{ij}^{(k)}$ are periodic functions of t with the period 2π, and where not all the $\theta_{ij}^{(0)}$ are constants. For $\mu = 0$ these equations reduce to

$$(x_i^{(0)})' = \sum_{j=1}^{n} \theta_{ij}^{(0)} x_j^{(0)}, \qquad i = 1, \ldots, n. \tag{81}$$

The solutions of these equations in general have the form

$$x_i^{(0)} = \sum_{j=1}^{n} A_j^{(0)} e^{\alpha_j^{(0)}(t)} y_{ij}^{(0)}(t), \qquad i = 1, \ldots, n, \tag{82}$$

where the $y_{ij}^{(0)}$ are periodic functions of t with the period 2π.

Suppose the $\alpha_j^{(0)}$ and $y_{ij}^{(0)}$ are fully known. We desire the solutions of (80). It follows from the general results of § 3 that there is at least one solution of the form

$$x_i = e^{\alpha t} y_i, \qquad i = 1, \ldots, n, \tag{83}$$

where the y_i are periodic with the period 2π. Now equations (80) can be integrated as power series in μ, reducing to $x_i^{(0)}$ for $\mu = 0$. We form n solutions, $x_{i1}, \ldots, x_{in}$, defined by the initial conditions $x_{i\kappa}^{(0)} = 0$, $x_{\kappa\kappa}^{(0)} = 1$, $\kappa = 1, \ldots, n$. Then any solution can be expressed linearly and homogeneously in terms of these n solutions. Hence

$$x_i = \sum_{j=1}^{n} A_j x_{ij}, \qquad i = 1, \ldots, n. \tag{84}$$

Transforming (80) by (83), we get

$$y_i' + \alpha y_i = \sum_{j=1}^{n} \left[\sum_{k=0}^{\infty} \theta_{ij}^{(k)} \mu^k \right] y_j, \qquad i = 1, \ldots, n.$$

Consequently the conditions that the y_i shall be periodic with the period 2π are $y_i(2\pi) - y_i(0) = 0$, $i = 1, \ldots, n$; or, because of (83), (84) and the initial conditions imposed on the x_{ij},

$$\sum_{j=1}^{n} A_j \left[e^{-2\alpha\pi} x_{ij}(2\pi) - x_{ij}(0) \right] = 0, \qquad i = 1, \ldots, n. \tag{85}$$

Now, since $x_{ij} = \sum_{k=0} x_{ij}^{(k)} \mu^k$, the determinant of the coefficients of the A_j set equal to zero is

$$\Delta = \left| e^{-2\alpha\pi} x_{ij}^{(0)}(2\pi) - x_{ij}^{(0)}(0) + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k \right| = 0. \tag{86}$$

This is an equation for the determination of α. For $\mu = 0$ the solutions are $\alpha = \alpha_j^{(0)} + \nu \sqrt{-1}$, where ν is any integer. Consequently, if $\alpha_\kappa^{(0)}$ is a simple

root of (86) and if no two of the $\alpha_j^{(0)}$ differ by an imaginary integer, then (86) can be solved for $(\alpha - \alpha_\kappa^{(0)})$ as a converging power series in μ. The results substituted in (85) give the ratios of the A_j as power series in μ for $\kappa = 1, \ldots, n$. When these results are substituted in (84), we have the solutions expanded as converging power series in μ, and they have the form (83), where α and the y_i are power series in μ, and where the latter are periodic with the period 2π.

There are other cases where for $\mu = 0$ the roots of (86) satisfy different conditions. The discussion is parallel to that where for $\mu = 0$ the θ_{ij} are constants. The essentials are that the differential equations shall be expansible as power series in a parameter μ, and that for $\mu = 0$ the solutions shall be known. The process is fundamentally one of analytic continuation of the solutions with respect to the parameter μ, and it can be repeated and continued from one value of μ to any other, provided the series do not pass through a singularity in the interval.

NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.*

§ 14. *Case where the Right Members Are Periodic with the Period 2π and the α_j Are Distinct.*

The problems of celestial mechanics generally lead to sets of differential equations having right members which are functions of the independent variable alone. The character of the terms in the solutions introduced by these right members depends not only upon the properties of the latter but also upon the α_j. We shall treat the most useful cases.

Suppose

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = g_i(t), \qquad i = 1, \ldots, n, \tag{87}$$

where the θ_{ij} and g_i are finite and continuous periodic functions of t having the period 2π. For the left members set equal to zero the general solutions are in this case

$$x_i = \sum_{j=1}^{n} \eta_j e^{\alpha_j t} y_{ij}, \qquad i = 1, \ldots, n, \tag{88}$$

where the y_{ij} are periodic with the period 2π and the η_j are the constants of integration.

* For a different treatment of this subject see paper by W. D. MACMILLAN, *Transactions of American Mathematical Society*, Vol. XI, No. 1, p. 85.

By the method of the variation of parameters we find

$$\sum_{j=1}^{n} \eta_j' e^{\alpha_j t} y_{ij} = g_i(t), \qquad i = 1, \ldots, n. \tag{89}$$

The determinant of the coefficients of the η_j' is the determinant of the fundamental set of solutions, and can vanish for no value of t for which the θ_{ij} are regular (§ 2). By hypothesis the θ_{ij} are regular for all finite values of t. Therefore this determinant can not vanish, and is

$$e^{\sum_{j=1}^{n} \alpha_j t} \Delta,$$

where Δ is the determinant of the y_{ij}. Consequently the solutions of (89) have the form

$$\eta_j' = e^{-\alpha_j t} \frac{\Delta_j}{\Delta}, \tag{90}$$

where Δ_j and Δ are periodic functions of t with the period 2π, and moreover Δ vanishes for no finite value of t. Hence Δ_j/Δ can be expanded as Fourier series of the form

$$\frac{\Delta_j}{\Delta} = a_0^{(j)} + \sum_{m=1}^{\infty} [a_m^{(j)} \cos mt + b_m^{(j)} \sin mt].$$

Since the Δ_j and Δ are power series in μ and in general Δ does not vanish for $\mu = 0$, this result can also be arranged as a power series in μ whose coefficients are periodic with the period 2π, but it is more convenient here to regard it simply as a Fourier series.

If $\alpha_j^2 + m^2 \neq 0$, $j = 1, \ldots, n$, $m = 0, \ldots, \infty$, we have

$$\eta_j = \int e^{-\alpha_j t} \frac{\Delta_j}{\Delta} dt = -\frac{a_0^{(j)}}{\alpha_j} e^{-\alpha_j t} + e^{-\alpha_j t} \sum_{m=1}^{\infty} \left[-\frac{\alpha_j a_m^{(j)} + m b_m^{(j)}}{\alpha_j^2 + m^2} \cos mt + \frac{m a_m^{(j)} - \alpha_j b_m^{(j)}}{\alpha_j^2 + m^2} \sin mt \right] + B_j;$$

or,

$$\eta_j = B_j + e^{-\alpha_j t} P_j(t), \tag{91}$$

where the $P_j(t)$ are periodic with the period 2π and the B_j are constants of integration. Substituting in (88), we get

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + \sum_{j=1}^{n} y_{ij} P_j(t), \qquad i = 1, \ldots, n, \tag{92}$$

as the general solutions of (87), the $P_j(t)$ being periodic with the period 2π. The terms $y_{ij} P_j(t)$ are the ones due to the presence of the $g_i(t)$.

Now let us suppose that $\alpha_k = \nu\sqrt{-1}$, where ν is an integer. Then the term

$$\int e^{-\nu\sqrt{-1}t}\left[a_\nu \cos \nu t + b_\nu \sin \nu t\right] dt$$

becomes

$$\tfrac{1}{2}(a_\nu - b_\nu\sqrt{-1})\, t + \frac{1}{4\nu}(a_\nu\sqrt{-1} - b_\nu)(\cos 2\nu t - \sqrt{-1}\sin 2\nu t).$$

Hence in this case we get

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + \sum_{j=1}^{n} y_{ij} P_j(t) + \tfrac{1}{2}(a_\nu - b_\nu\sqrt{-1})\, t\, e^{\alpha_\kappa t} y_{i\kappa}, \quad i = 1, \ldots, n. \tag{93}$$

Hence, *if the α_j are distinct and none of them congruent to zero mod $\sqrt{-1}$, and if the $g_i(t)$ are periodic with the period 2π, then the particular integrals are also periodic with the period 2π; but if one of the characteristic exponents is congruent to zero mod $\sqrt{-1}$, then the particular integrals in general will contain, in addition to periodic terms, the corresponding parts of the complementary function multiplied by a constant times t.*

§ 15. *Case where the Right Members Are Periodic Terms Multiplied by an Exponential, and the α_j Are Distinct.*

We now suppose the $g_i(t)$ have the form

$$g_i(t) = e^{\lambda t} f_i(t), \tag{94}$$

where the $f_i(t)$ are periodic with the period 2π. When λ is a pure imaginary, $\lambda = l\sqrt{-1}$, the $g_i(t)$ have the form

$$g_i(t) = \sum_{k=0}^{\infty}\left[a_k \cos(k+l)\,t + b_k \sin(k+l)\,t\right],$$

which appears often in applications to celestial mechanics.

If we transform the differential equations

$$x_i - \sum_{j=1}^{n} \theta_{ij} x_j = e^{\lambda t} f_i(t) \tag{95}$$

by

$$x_i = e^{\lambda t} z_i,$$

we obtain

$$z_i' + \lambda z_i - \sum_{j=1}^{n} \theta_{ij} z_i = f_i(t), \qquad i = 1, \ldots, n, \tag{96}$$

which have the same character as the equations treated in § 14. If the α_j are distinct, then the characteristic exponents $\alpha_j - \lambda$ belonging to (96) are also

distinct. If no $\alpha_j - \lambda$ is congruent to zero mod $\sqrt{-1}$, then the general solutions of (96) are

$$z_i = B_j e^{(\alpha_j - \lambda) t} y_{ij} + Q_i(t),$$

where the $Q_i(t)$ are periodic with the period 2π.

Therefore in this case the general solutions of (95) are

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + e^{\lambda t} Q_i(t), \qquad i = 1, \ldots, n. \tag{97}$$

But if λ is congruent to one of the α_j, say α_ν, mod $\sqrt{-1}$, then the z_i have the form

$$z_i = \sum_{j=1}^{n} B_j e^{(\alpha_j - \lambda) t} y_{ij} + \sum_{j=1}^{n} y_{ij} P_j(t) + \tfrac{1}{2}(a_\nu - b_\nu \sqrt{-1})\, t e^{(\alpha_\nu - \lambda) t} y_{i\nu},$$

and therefore

$$x_i = \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + e^{\lambda t} \sum_{j=1}^{n} y_{ij} P_j(t) + \tfrac{1}{2}(a_\nu - b_\nu \sqrt{-1})\, t e^{\alpha_\nu t} y_{i\nu}. \tag{98}$$

Therefore, *if the $g_i(t)$ are $e^{\lambda t}$ times periodic functions and if no $\alpha_j - \lambda$ is congruent to zero mod $\sqrt{-1}$, then the particular solution is $e^{\lambda t}$ times a periodic function; but if $\alpha_\nu - \lambda$ is congruent to zero mod $\sqrt{-1}$, then the particular solution is $e^{\lambda t}$ times a periodic function plus a constant times $t e^{\alpha_\nu t} y_{i\nu}$.*

§ 16. *Case where the Right Members Are Periodic and $\alpha_2 = \alpha_1$.*

In case there are no equalities among the α_j except $\alpha_2 = \alpha_1$, the solutions of

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = 0, \qquad i = 1, \ldots, n,$$

in general have the form

$$x_i = \eta_1 e^{\alpha_1 t} y_{i1} + \eta_2 e^{\alpha_1 t} (y_{i2} + t y_{i1}) + \sum_{j=3}^{n} \eta_j e^{\alpha_j t} y_{ij}, \qquad i = 1, \ldots, n. \tag{99}$$

For the non-homogeneous equations

$$x_i' - \sum_{j=1}^{n} \theta_{ij} x_j = g_i(t)$$

we find, by the variation of parameters,

$$e^{\alpha_1 t} y_{i1} \eta_1' + e^{\alpha_1 t} (y_{i2} + t y_{i1}) \eta_2' + \sum_{j=3}^{n} e^{\alpha_j t} y \ \eta_j' = g_i(t), \qquad i = 1, \ldots, n.$$

Solving these equations for the η_j', we get

$$\Delta\eta_1' = |\, g_i(t),\ (y_{i2}+ty_{i1}),\ y_{i3},\ \ldots,\ y_{in}\,|\, e^{-\alpha_1 t},$$
$$\Delta\eta_2' = |\, y_{i1},\ \ g_i(t),\ \ y_{i3},\ \ldots,\ y_{in}\,|\, e^{-\alpha_1 t},$$
$$\Delta\eta_j' = |\, y_{i1},\ (y_{i2}+ty_{i1}),\ y_{i3},\ \ldots,\ y_{in}\,|\, e^{-\alpha_j t},\quad j=3,\ \ldots,\ n,$$

where Δ is the determinant $|y_{ij}|$. The expansions of these determinants have the form

$$\left.\begin{aligned} \eta_1' &= e^{-\alpha_1 t}P_1(t) - e^{-\alpha_1 t}t\,P_2(t),\\ \eta_2' &= e^{-\alpha_1 t}P_2(t),\\ \eta_j' &= e^{-\alpha_j t}P_j(t), \qquad j=3,\ \ldots,\ n, \end{aligned}\right\} \tag{100}$$

where the $P_1(t),\ \ldots,\ P_n(t)$ are periodic with the period 2π.

If no α_j is congruent to zero mod $\sqrt{-1}$, then we find

$$\left.\begin{aligned} \eta_1 &= e^{-\alpha_1 t}R_1(t) - e^{-\alpha_1 t}t\,R_2(t) + B_1,\\ \eta &= e^{-\alpha_1 t}R_2(t) + B_2,\\ \eta_j &= e^{-\alpha_j t}R_j(t) + B_j, \qquad j=3,\ \ldots,\ n, \end{aligned}\right\} \tag{101}$$

where $R_1,\ \ldots,\ R_n$ are periodic with the period 2π. Substituting (101) in (99), we get

$$x_i = B_1 e^{\alpha_1 t}y_{i1} + B_2 e^{\alpha_1 t}(y_{i2}+ty_{i1}) + \sum_{j=3}^{n} B_j e^{\alpha_j t}y_{ij} + \sum_{j=1}^{n} R_j y_{ij}. \tag{102}$$

The terms introduced into the solutions by the right members of the differential equations are $\sum_{j=1}^{n} R_j y_{ij}$, which are periodic. Therefore *it follows that equalities among the characteristic exponents, without congruences to zero mod* $\sqrt{-1}$, *do not introduce non-periodic terms into this part of the solutions.*

The case where one α_j, $j=3,\ \ldots,\ n$, is congruent to zero mod $\sqrt{-1}$, is a combination of the second part of §14 and this case; and that where $\alpha_2=\alpha$ is congruent to zero mod $\sqrt{-1}$ does not differ from that where $\alpha_2=\alpha_1=0$.

Consequently we consider the case $\alpha_2=\alpha_1=0$, which frequently arises in celestial mechanics. Then the equations corresponding to (100) become

$$\left.\begin{aligned} \eta_1' &= P_1(t) - t\,P_2(t),\\ \eta_2' &= P_2(t),\\ \eta_j' &= e^{-\alpha_j t}P_j(t), \qquad j=3,\ \ldots,\ n, \end{aligned}\right\} \tag{103}$$

where

$$P_j = \sum_{k=0}^{\infty}\left[a_k^{(j)}\cos kt + b_k^{(j)}\sin kt\right], \qquad j=1,\ \ldots,\ n.$$

Hence we find

$$\left.\begin{aligned}\eta_1 &= a_0^{(1)}t - \tfrac{1}{2}a_0^{(2)}t^2 - tR_2(t) + R_1(t) + B_1,\\ \eta_2 &= a_0^{(2)}t + R_2(t) + B_2,\\ \eta_j &= e^{-\alpha_j t}R_j(t) + B_j, \qquad j = 3, \ldots, n,\end{aligned}\right\} \quad (104)$$

where the $R_j(t)$ are periodic with the period 2π. Substituting these values in (99), we get

$$x_i = B_1 y_{i1} + B_2(y_{i2} + t y_{i1}) + \sum_{j=1}^{n} B_j e^{\alpha_j t} y_{ij} + (a_0^{(1)}t + \tfrac{1}{2}a_0^{(2)}t^2)\, y_{i1} + a_0^{(2)} t y_{i2} + \sum_{j=3}^{n} R_j(t)\, y_{ij}. \qquad (105)$$

Hence, *when the $g_i(t)$ are periodic with the period 2π, and two of the α_j are not only equal but also equal to zero, then the particular integral involves not only t but, in general, also t^2 outside of the trigonometric symbols.* It can be shown similarly that when k of the α_j are equal to zero, then in general the solutions are polynomials in t of degree k whose coefficients are periodic functions of t.

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